Detecting cyclical asymmetry: The filter matters*

[Version 1.0.5]

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March 15, 2020

Abstract

In this paper I revisit the role of detrending in detecting cyclical asymmetry. Psaradakis and Sola (*Journal of Applied Econometrics* 2003; **18**(3)): 271-289) show that detrending using the Hodrick-Prescott filter, Baxter-King bandpass filter, and Beveridge-Nelson decomposition result in asymmetry test statistics with poor power properties which are biased in favor of symmetry. I first confirm this result and second show that one-sided regression-based filters lead to much better power properties relative to the Hodrick-Prescott and Baxter-King filters. In an application to US output and hours worked, the conclusions regarding cyclical asymmetry are different depending on the type filter used.

1 Introduction

Since most macroeconomic time series are non-stationary due to factors such as secular growth, analyzing business cycle fluctuations requires separating the non-stationary trend component from the stationary (or business cycle) fluctuations. This is primarily done using detrending techniques. However, detrending techniques can often distort aspects of the underlying cyclical component. This paper is concerned with distortions to any potential asymmetry in the business cycle component.

Psaradakis and Sola (2003) (henceforth PS) show that unless the underlying cyclical component is manifestly asymmetric, tests for asymmetry have little power after detrending. They

*First draft: March 2, 2020. Thanks to Hashmat Khan, James Morley, and Jean-François Rouillard for comments on previous versions of this paper.

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investigate the properties of the Hodrick-Prescott (HP) filter, Baxter-King (BK) bandpass filter, and the traditional Beveridge-Nelson decomposition. While these filters are arguably the most common methods for detrending time series in macroeconomics, they have many documented shortcomings (c.f., Kamber et al. (2018); Hamilton (2018)). In this paper I confirm the results in PS regarding poor power properties after detrending. In addition I show that relative to the HP and BK filters, one-sided regression-based filters lead to much better power properties in a standard asymmetry test.

In this paper I assess the impact of the Hodrick-Prescott (HP) filter (Hodrick and Prescott (1980, 1997)) and Baxter-King (BK) bandpass filter (Baxter and King (1999)), and compare with the recently developed Hamilton filter (Hamilton (2018)), and Beveridge-Nelson (BN) filter (Kamber et al. (2018)), on the power to detect business cycle asymmetry (i.e., skewness in the distribution of cyclical fluctuations).[1]

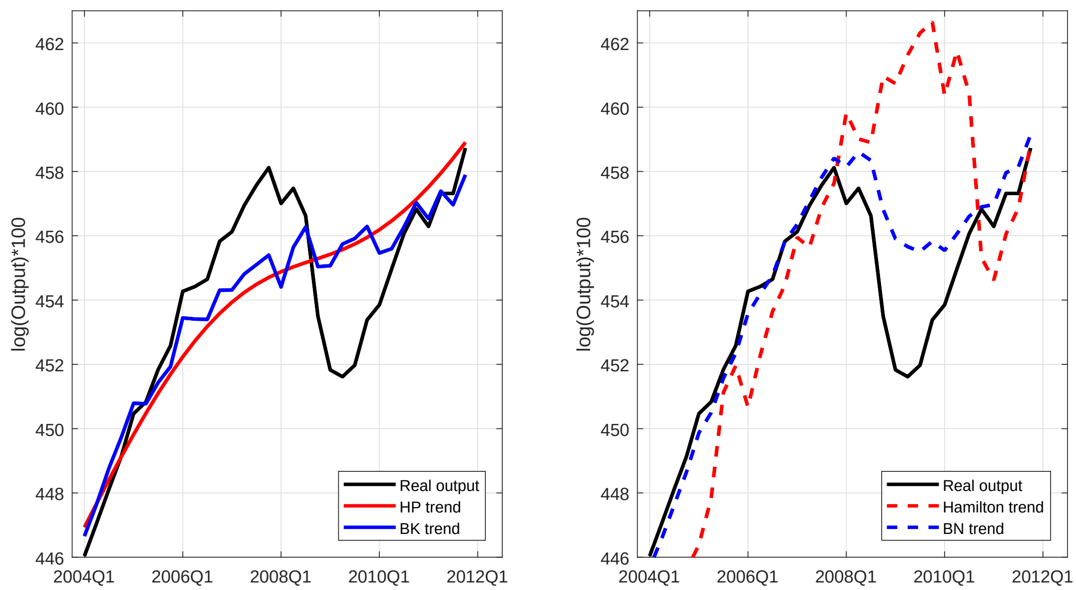
I show that two-sided detrending techniques (detrending methods that make use of past and future information to estimate the trend), such as the HP filter and BK bandpass filter, tend to smooth out any asymmetry that may be present in the underlying cyclical component. Testing for asymmetry in the cyclical component will then be strongly biased in favor of symmetry. In contrast, the Hamilton filter and BN filter are one-sided regression-based filters (detrending methods which only make use of past information to estimate the trend). Both of these filters are much better at preserving any asymmetries in the underlying cyclical component.

To illustrate this point I report a simulation exercise in which I generate non-stationary time series based on a unobserved components model estimated on US real GDP. I then vary the degree of asymmetry in the underlying cyclical component and test for asymmetry after detrending. While all filters lead to lower power properties than the true (unobserved in practice) cyclical component, the two-sided filters (HP & BK) considered here perform poorly and do not result in improved power properties when either the sample size is increased or the asymmetry in the underlying

[1] I explored the traditional Beveridge-Nelson decomposition, but its estimates of cyclical fluctuations in the post-1982 period were wildly at odds with the rest of the filters considered. While the traditional BN decomposition performs better than HP and BK filters in terms of preserving asymmetry, as noted by PS, it still performs significantly worse than the Hamilton and BN filters. These results are available upon request.

cyclical component is increased. In contrast, the Hamilton and BN filters report much better power properties in general, and these properties improve when the sample size is increased or the degree of asymmetry in the cyclical component is increased.

Figure 1: Trend estimates over the Great Recession



Notes: Output is the non-farm business sector real output (FRED code: OUTNFB). The HP trend is estimated using a smoothing parameter of 1600. The BK bandpass trend is estimated isolating frequencies between 6 and 32 quarters. The Hamilton filters uses 4 lags and forecasts 8 quarters into the horizon. The BN trend is estimated using 12 lags and a signal-to-noise ratio of 0.85.

The rationale for why the HP and BK filters lead to such poor power properties in asymmetry test statistics is due to the fact that these two-sided filters tend to impart common behavior on booms and busts over the business cycle. This intuition is illustrated in Figure 1, which plots the log of quarterly US real non-farm output over the Great Recession, along with the corresponding trend estimates from the HP & BK filters in the left panel and the Hamilton & BN filters in the right panel. Since the HP and BK filters make use of future information, there are significant movements in the trend prior to the onset of the recession. This results in a more significant estimated boom and less significant estimated bust (in terms of deviation from trend). In contrast, since the Hamilton and BN filters trend estimates are purely backward looking, the trend does not decline at all prior to the recession. The resulting cyclical component features a much smaller boom and much larger contraction than in the case of the HP or BK filters.

The implications of using a one-sided filter versus two-filter is not merely of qualitative significance. In an application to US output and hours worked, I show that the conclusions about cyclical asymmetry in the post-Great Moderation period differ depending on the type of filter used. Specifically, testing for asymmetry after detrending using the HP or BK filters, I do not reject the null hypothesis of symmetric cyclical fluctuations for output or hours worked. In contrast, after detrending using the Hamilton or BN filter I reject the null hypothesis for both output and hours worked.

The rest of the paper proceeds as follows. In Section 2 I describe the Hamilton and BN filters as well as the asymmetry test used in the paper. Section 3 implements a simulation study in which I compare the ability to detect cyclical asymmetry after detrending using the various filters. Section 4 provides an application to US output and hours worked in the Great Moderation period and contrasts the results offered via one-sided filters and two-sided filters. Section 5 concludes.

2 Detrending methods & skewness test

In the following section I describe the Hamilton and BN filters, as well as the asymmetry test used in the paper. While I also report results in the paper for the HP and BK filters, the techniques are common enough that I do not discuss the methodology for these filters (see Hodrick and Prescott (1980, 1997) and Baxter and King (1999)). To test for cyclical asymmetry I use a standard test in the macroeconomic literature which amounts to hypothesis testing on the coefficient of skewness in the cyclical component (Sichel (1993)).

2.1 Hamilton filter

Recently, Hamilton (2018) has proposed an alternative to the HP filter which I will refer to as the *Hamilton* filter. The proposed approach overcomes several well documented shortcomings of the original HP filter, including: the HP filter being inappropriate for time series featuring random walks and the challenge of choosing an appropriate smoothing parameter. In contrast, the Hamilton filter offers a reasonable approach to construct the cyclical component for a wide variety of data generating processes.

The Hamilton filter requires obtaining residuals from a regression of a variable h periods ahead based on its p most recent values as of date t. Specifically for a univariate series, y_t, I run the following regression,

$$y_{t+h} = \beta_0 + \sum_{j=0}^{p} \beta_{j+1} y_{t-j} + c_{t+h}, \tag{1}$$

and construct the cyclical component as the residual given by,

$$\hat{c}_{t+h} = y_{t+h} - \underbrace{\left(\hat{\beta}_0 + \sum_{j=0}^{p} \hat{\beta}_{j+1} y_{t-j}\right)}_{\hat{\tau}_{t+h}}. \tag{2}$$

In the implementation of the Hamilton filter, I use $h = 8$ and $p = 4$ which are Hamilton (2018)'s suggested parametric specification for detrending quarterly data. Since the filter only uses 4 lags and a constant to estimate the trend, it is one-sided.[2]

2.1.1 Hamilton random walk filter

The Hamilton filter is appropriate for a wide variety of data generating processes. If the series is characterized by a random walk with a large enough sample, the estimates for β_1 will converge to 1 and all other β_j to 0. However, in practice with short samples, these coefficients may be different from their asymptotic limits (i.e., not equal to 0). Since the data generating process in the simulation exercise is characterized by a random walk, I also report the results for the random

[2] The OLS regression does make use of future observations in the estimation of the regression coefficients, however as (Hamilton, 2018, pg. 837) notes, this influence vanishes asymptotically.

walk version of the Hamilton filter.[3] More explicitly, the cyclical component for the random walk Hamilton filter can be constructed as the follows,

$$c_{t+h} = y_{t+h} - \underbrace{y_t}_{\hat{\tau}_{t+h}}, \tag{3}$$

where the forecast horizon is again 8 quarters ($h = 8$).

2.2 Modified Beveride-Nelson Decomposition

A recent innovation related to the Beveridge-Nelson decomposition has been proposed by Kamber et al. (2018), who note that output gap measures based on the traditional Beveridge-Nelson decomposition looks quite different from other intuitive measures of the output gap (such as those produced by the Congressional Budget Office (CBO) or the NBER reference cycle). Estimates from the Beveridge-Nelson decomposition have small amplitude and little persistence. The reason is that the decomposition implies a high signal-to-noise ratio (i.e., the variance of changes in the trend is large relative to the variance of the forecast errors).

Kamber et al. (2018) propose a modification of the Beveridge-Nelson decomposition which they refer to as the *BN filter*. The modification involves imposing a lower signal-to-noise ratio by fixing the sum of the autoregressive coefficients in the following AR(p) model,

$$\Delta y_t = \rho \Delta y_{t-1} + \sum_{j=1}^{p-1} \phi_j \Delta^2 y_{t-j} + e_t, \tag{4}$$

where Δy_t is the demeaned first difference of the log of the series and $\rho \equiv \phi_1 + \phi_2 \cdots + \phi_p$. After obtaining estimates for the parameters, the cyclical component can be constructed using the state space approach in Morley (2002),

[3] In the simulation exercise and the empirical application, the results between the random walk filter and the original Hamilton filter are very similar.

$$c_t = [-1\ 0\ \ldots\ 0]F(I-F)^{-1}X_t, \tag{5}$$

where F is the companion form matrix containing estimates, $\hat{\phi}_j$, and X_t contains current and lagged values of Δy_t.

To impose a lower signal-to-noise ratio involves fixing $\bar{\rho} = 1 - 1/\sqrt{\bar{\delta}}$, where δ is the signal-to-noise ratio. Similar to the Hamilton filter, since the regression uses only lagged values, it is one-sided. To implement the BN filter I follow Kamber et al. (2018) and set the lag length equal to 12. To choose the appropriate signal-to-noise ratio, I use the maximization routine provided by the authors which maximizes the amplitude-to-noise ratio for signal-to-noise ratios closest to 0.

2.3 Skewness test

To test for business cycle asymmetry I implement a standard test used in the literature which amounts to hypothesis testing on the coefficient of skewness in the estimated cyclical component (see, Sichel (1993)). The coefficient of skewness is given by,

$$D(c) = \frac{\frac{1}{T}\sum_{t=1}^{T}(c_t - \bar{c})^3}{\sigma(c)^3}, \tag{6}$$

where c_t, $\bar{c}$, and $\sigma(c)$ represent the cyclical component, the mean of the cyclical component, and the standard deviation of the cyclical component, respectively. Since the time series in the simulation and empirical application are both characterized by serial correlation, standard critical values do not apply. For appropriate critical values I use the methods outlined in Bai and Ng (2005) who show that by using Newey-West corrected standard errors and adding a normalizing constant to the standard error proportional to the standard deviation of the cyclical component, the limiting distribution is normal with unit variance.

While the hypothesis test can be formulated as a two-tailed test, it is generally accepted that if the business cycle is asymmetric, it is characterized by negative skewness. Sichel (1993) refers to this type of asymmetry as *Deepness*, which describes cyclical fluctuations where the trough of the recession is further from trend (in absolute value) than peaks are above trend.[4] Thus the hypothesis is formulated as a one-tailed test with the corresponding null and alternative hypotheses given by,

$$H_0 : D(c) \geq 0, \tag{7}$$

$$H_A : D(c) < 0. \tag{8}$$

The above states that if the null hypothesis is not rejected, we cannot reject that business cycle fluctuations are symmetric (more accurately, they are not negatively skewed). Conversely, if the null hypothesis is rejected, there is evidence of deepness and business cycle asymmetry.

3 A simulation study

To assess the impact of different filtering techniques on the ability to detect cyclical asymmetry I perform a simulation study. Specifically, I generate non-stationary time series based on the following unobserved components model,

$$\tau_t = 0.01 + \tau_{t-1} + 0.01\eta_t, \tag{Trend component}$$

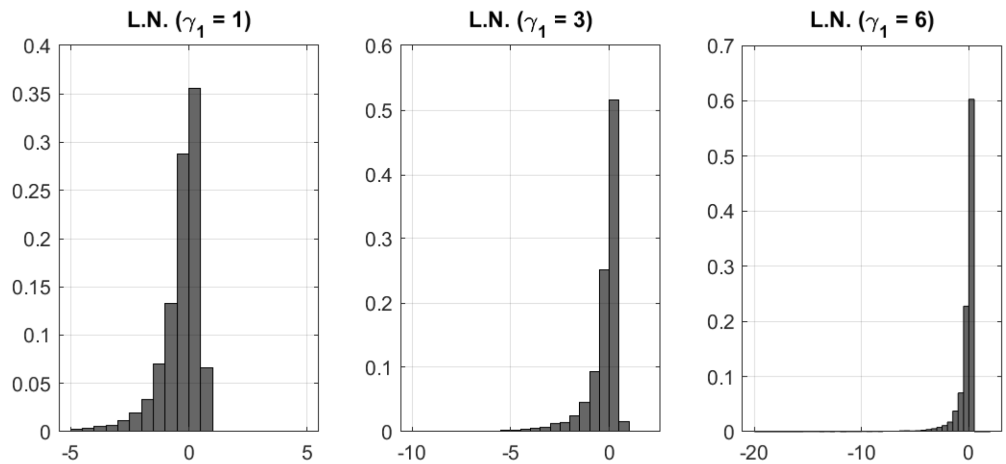
$$c_t = 1.50c_{t-1} - 0.52c_{t-2} + 0.01u_t, \tag{Cyclical component}$$

$$y_t = \tau_t + c_t, \tag{Non-stationary output series}$$

[4] Alternatively, as noted by Hansen and Prescott (2005), one could think of this asymmetry as the average deviation below trend being larger (in absolute value) than the average deviation above trend.

where the trend component is characterized by a unit root process with a drift.[5] η_t are i.i.d. $N(0,1)$ random variables and u_t is drawn from the following distributions: (1) a normal distribution with mean 0 and variance equal to 1; (2) a log normal distribution with variance equal to 1 (mildly asymmetric); (3) a log normal distribution with variance equal to 3 (moderately asymmetric); (4) a log normal distribution with variance equal to 6 (strongly asymmetric). All distributions are rescaled and recentered to have zero mean and standard deviation of one. The asymmetric distributions (after rescaling) are depicted in the Figure 2.

Figure 2: Error distributions used in the simulation study



I generate 1000 series of y_t for each type of cyclical error distribution with length 152 and 312.[6] These lengths are chosen to represent the empirical sample size in the post Great Moderation sample used in the empirical application and the sample size of the entire post-WWII data. After generating samples of non-stationary data I detrend the data using the HP, BK, Hamilton, and BN filters and compute the asymmetry test statistics.

Table 1 reports the rejection frequencies of the asymmetry test at the 5% level across different error distributions and filters. The first column y_t is the true (though unobserved in practice)

[5]Similar approaches to modeling US output were taken in Watson (1986) and Morley et al. (2003). In the Appendix I describe the state space setup and estimation of the model.

[6]Since the Hamilton filter forecasts 8 periods ahead based on the 4 most recent values, 12 missing observations are created. To facilitate comparison across the filters, I drop 12 observations from each filtered series. The resulting series contain 140 and 300 observations, respectively. To ensure that initial conditions play no role, I use a 300 period burn-in.

cyclical component.

Table 1: SIMULATION REJECTION FREQUENCIES AT THE 5% LEVEL

Innovations	y_t	$\hat{y}_t^{HP}$	$\hat{y}_t^{BK}$	$\hat{y}_t^{hp}$	$\hat{y}_t^{hp_{RW}}$	$\hat{y}_t^{BN}$
			T = 140			
Normal	0.134	0.059	0.058	0.098	0.112	0.084
L.N. ($\gamma_1 = 1$)	0.445	0.108	0.062	0.327	0.341	0.381
L.N. ($\gamma_1 = 3$)	0.599	0.094	0.042	0.493	0.500	0.528
L.N. ($\gamma_1 = 6$)	0.685	0.116	0.030	0.563	0.581	0.577
			T = 300			
Normal	0.179	0.068	0.047	0.084	0.088	0.080
L.N. ($\gamma_1 = 1$)	0.548	0.110	0.039	0.418	0.379	0.489
L.N. ($\gamma_1 = 3$)	0.811	0.104	0.021	0.575	0.540	0.617
L.N. ($\gamma_1 = 6$)	0.874	0.094	0.020	0.597	0.592	0.615

Notes: y_t^{HP}, y_t^{BK}, y_t^{hp}, $y_t^{hp_{RW}}$, y_t^{BN} denote cyclical components obtained from the HP filter, Baxter-King bandpass filter, Hamilton filter, Hamilton random walk filter, and BN filter, respectively. y_t is the true cyclical process (empirically unobserved).

The simulation exercise reveals two points worth emphasizing with respect to detrending. First, relative to the true unobserved cyclical component, all filters considered reduce the power of detecting deepness. This point was emphasized by Psaradakis and Sola (2003). However, this reduction in power is not uniform across filters. The two-sided filters (HP & BK) perform significantly worse than the one-sided filters.

Second, even when the underlying cyclical component is significantly asymmetric (e.g., in the case of $\gamma_1 = 6$), the HP and BK filters power properties do not improve. This suggests that the filter itself tends to impart a common behavior on booms and busts which leads to low power properties in the asymmetry test. This is reinforced by the fact that even when the sample size is increased from $T = 140$ to $T = 300$, the bias in filtering remains. This result is perhaps not surprising given the intuition in the introduction.

Relative to the two-sided HP and BK filters, the Hamilton and BN filters display much more satisfactory properties. In general they perform better at preserving any underlying cyclical asymmetry, as shown by the fact that the rejection rates after detrending using these methods are closer to the true (unobserved) cyclical component. Additionally, these filters lead to better power properties when either the magnitude of asymmetry is increased or the sample size is increased.

The simulation results show that one-sided filters lead to much better power properties of asymmetry test statistics and represent a substantial improvement over their two-sided alternatives.

4 Application to US output and hours worked

To highlight that filtering differences have important consequences for conclusions about asymmetry over the business cycle, I apply the skewness test described in Section 2 to US real non-farm output and hours worked.[7] The sample contains quarterly data from 1982Q1:2019Q3. The rationale for focusing on this sample period is that the behavior of the US business cycle after 1982 is distinctly different from the pre-1982 cycles in terms of expansion length and cyclical fluctuations. Both output and hours worked are logged and detrended using the HP, BK, Hamilton, and BN filters.

Table 2 reports skewness coefficients, test statistics, and p-values for one-sided hypothesis tests that output and hours worked are characterized by deepness over the business cycle.

Both output and hours worked tell a similar story. After detrending using the HP or BK filters there is little evidence for asymmetry in the cyclical component. Sample skewness coefficients are small and not statistically different from zero. In contrast, after detrending using the Hamilton or BN filters, the asymmetry tests suggest strong evidence of cyclical asymmetry.[8] Sample skewness coefficients are large and the null hypothesis is rejected, often below the 5% critical value.

These results are consistent with the conclusions reached in the simulation exercise. If there is

[7]All data were retrieved from the Federal Reserve Economic Database (FRED). Output (FRED code: OUTNFB) is the real non-farm business sector output. Hours worked (FRED code: HOANBS) is the non-farm business sector hours worked.

[8]For the Hamilton filter, this is result holds regardless of whether it is assumed that output or hours worked contain a unit root.

Table 2: Deepness in US output and hours worked

	Output			
	$\hat{\mathbf{D}}(\mathbf{c})$	**Test statistic**	**P-value**	**Decision**
HP filter	-0.227	-0.700	0.242	Do not reject
Baxter-King filter	-0.296	-0.737	0.230	Do not reject
Hamilton filter	-1.083	-2.198	0.014	Reject
Hamilton RW filter	-0.843	-1.448	0.074	Reject
BN filter	-1.450	-1.552	0.060	Reject
	Hours worked			
	$\hat{\mathbf{D}}(\mathbf{c})$	**Test statistic**	**P-value**	**Decision**
HP filter	-0.215	-0.734	0.232	Do not reject
Baxter-King filter	-0.186	-0.473	0.318	Do not reject
Hamilton filter	-1.153	-2.388	0.009	Reject
Hamilton RW filter	-1.252	-2.279	0.011	Reject
BN filter	-1.436	-1.995	0.023	Reject

Notes: The null hypothesis for the test statistic is that the series exhibit no deepness ($H_0 : D(c) \geq 0$) and the alternative is that the series exhibits deepness ($H_A : D(c) < 0$). I use a smoothing parameter of $\lambda = 1600$ for the HP filter implementation. For the Baxter-King bandpass filter I isolate frequencies between 6 and 32 quarters. The Hamilton filter uses 4 lags ($p = 4$) and forecasts 8 periods ahead ($h = 8$). The BN filter optimization routine suggests a signal-to-noise ratio of 0.2 for output and 0.46 for hours. The decision cutoff is the 10% significance level.

asymmetry in the cyclical component, the two-sided nature of the HP and BK filters will tend to smooth out any asymmetry in the underlying cyclical component making it hard to test for these asymmetries after filtering. One-sided filters preserve asymmetry in the cyclical component and lead to much strong rejection rates in output and hours worked over in the post-Great Moderation period.

5 Conclusion

This paper has highlighted that one-sided regression-based filters are superior at preserving any underlying cyclical asymmetry relative to two-sided filters such as the Hodrick-Prescott filter and Baxter-King bandpass filter. In an application to US output and hours worked in the Great

Moderation period, I have shown that the type of filter matters. Specifically, there is little evidence of cyclical asymmetry in output or hours worked if one detrends the data using the HP or BK filters. But if one detrends using the Hamilton or BN filter, there is strong evidence of cyclical asymmetry. Simulation evidence suggests that results from the Hamilton and BN filters are more reliable, and by extension that output and hours worked are characterized by cyclical asymmetry.

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Online Appendix

A Simulation as a reduced-form ARIMA(2,1,2)

The simulation study features an unobserved trend, $\{\tau_t\}$, and unobserved stationary cycle, $\{c_t\}$, which yield an observable series, $\{y_t\}$,

$$\begin{aligned}\tau_t &= \mu + \tau_{t-1} + \eta_t,\\ c_t &= \phi_1 c_{t-1} + \phi_2 c_{t-2} + \epsilon_t,\\ y_t &= \tau_t + c_t.\end{aligned}$$

To see that this model has a univariate ARIMA(2,1,2) representation, note that the model can be written in its reduced form by the following,

$$y_t = \tau_t + c_t \rightarrow \ \Delta y_t = (1-L)\tau_t + (1-L)c_t$$

where $(1-L)\tau_t$ and c_t are given by,

$$\begin{aligned}(1-L)\tau_t &= \mu + \eta_t,\\ c_t &= (1-\phi_1 L - \phi_2 L^2)^{-1}\epsilon_t,\end{aligned}$$

which yields the following,

$$\Delta y_t = \mu + \eta_t + (1-L)(1-\phi_1 L - \phi_2 L^2)^{-1}\epsilon_t,$$

multiplying both sides by $(1-\phi_1 L - \phi_2 L^2)$ gives,

$$\Delta y_t(1-\phi_1 L - \phi_2 L^2) = (\mu + \eta_t)(1-\phi_1 L - \phi_2 L^2) + (1-L)\epsilon_t$$

which yields the reduced-form,

$$\Delta y_t = \mu^\star + \phi_1 \Delta y_{t-1} + \phi_2 \Delta y_{t-2} + u_t + \theta_1 u_{t-1} + \theta_2 u_{t-2},$$

where $\mu^\star = \mu(1-\phi_1-\phi_2)$, $u_t = \eta_t + \epsilon_t$, $\theta_1 u_{t-1} = \epsilon_{t-1} - \phi_1\eta_{t-1}$, and $\theta_2 u_{t-2} = -\phi_2\eta_{t-2}$.

A.1 Unobserved components model

The simulation study uses an unobserved components model setup given by,

$$\begin{aligned} \tau_t &= \mu + \tau_{t-1} + \eta_t, \\ c_t &= \phi_1 c_{t-1} + \phi_2 c_{t-2} + u_t, \\ y_t &= \tau_t + c_t. \end{aligned}$$

To estimate the model, it is useful to cast the unobserved components model in state space form,

$$\underbrace{\begin{bmatrix} \tau_t \\ c_t \\ c_{t-1} \end{bmatrix}}_{X_t} = \underbrace{\begin{bmatrix} \mu \\ 0 \\ 0 \end{bmatrix}}_{M} + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \phi_1 & \phi_2 \\ 0 & 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \tau_{t-1} \\ c_{t-1} \\ c_{t-2} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \eta_t \\ \epsilon_t \end{bmatrix}}_{u_t},$$

with the observation equation given by,

$$y_t = \underbrace{\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}}_{D} X_t.$$

In this state space setup it is straightforward to compute the log likelihood via the Kalman filter and estimate the parameters of the state space system (μ, ϕ_1, ϕ_2, σ_η, and σ_ϵ). Similar to Morley et al. (2003), I place two restrictions on the estimation. First, the sum of autoregressive parameters in the cyclical component must be stationary (i.e., $\phi_1 + \phi_2 \leq 1$). Second, I impose a positive definiteness constraint on the covariance innovation matrix.

The parameters used in the simulation are obtained by estimating this unobserved components model on non-farm US real GDP from 1948Q1:2016Q4. The results of the estimation are reported in Table 3,

Table 3: Maximum likelihood estimates of unobserved components model

Log-likelihood	$\hat{\mu}$	$\hat{\phi}_1$	$\hat{\phi}_2$	$\hat{\sigma}_\eta$	$\hat{\sigma}_\epsilon$
-358.08	0.800	1.516	-0.521	0.519	0.673